Solving differential equations on a superconducting quantum processor

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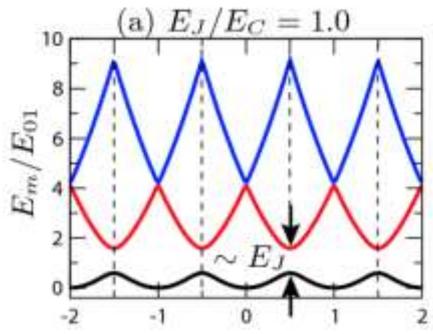
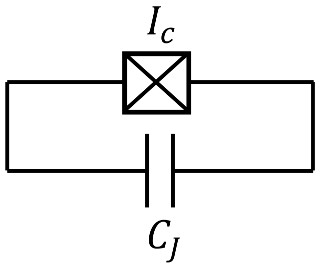
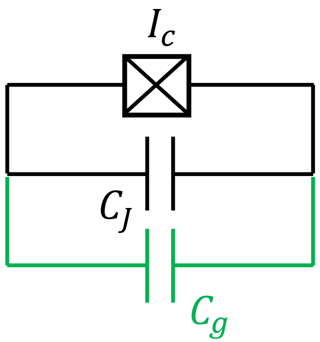
Development of quantum computational methods

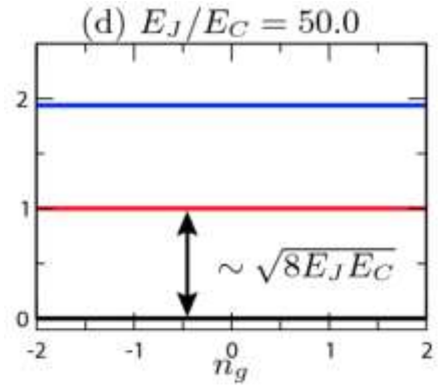
2008 : first unsensitive to background charge noise superconducting qubit is suggested (J. Koch et. al., 2008)



Cooper-pair box







2019-2023… - Noisy intermediate scale quantum (NISQ) devices epoch

- Quantum supremacy has been demonstrated in large size matrix multiplication (Google team)

Article

Quantum supremacy using a programmable superconducting processor

https://doi.org/10.1038/s41586-019-1666-5

Received: 22 July 2019

Accepted: 20 September 2019

Published online: 23 October 2019

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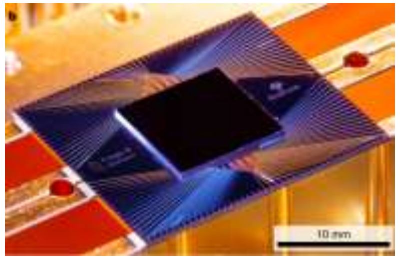


- First evidence of quantum simulation of spin dynamics has been demonstrated at a finite noise level of quantum gates (IBM team)

Article

Evidence for the utility of quantum computing before fault tolerance

https://doi.org/10.1038/s41586-023-06096-3

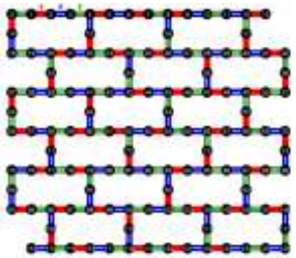
Received: 24 February 2023

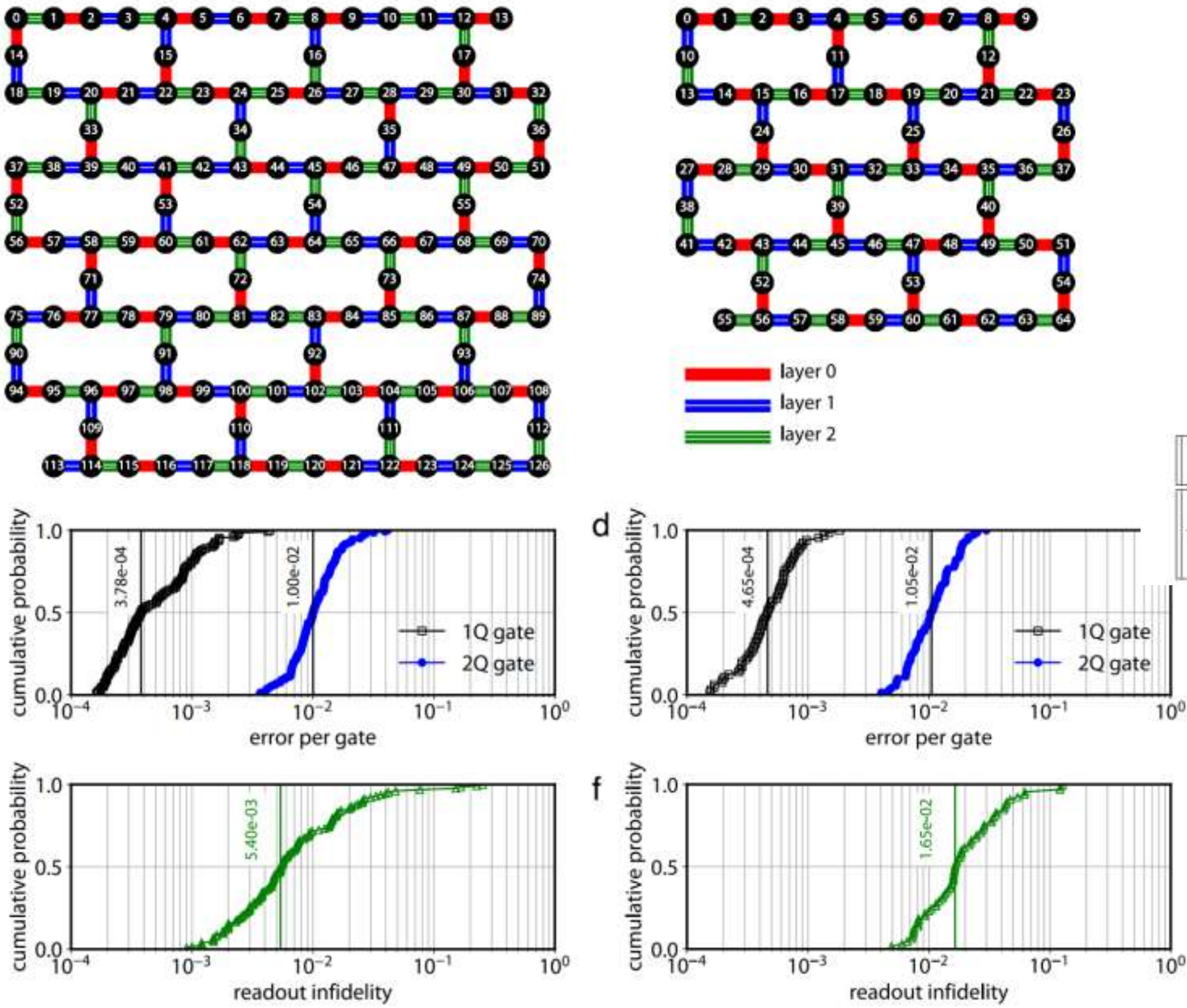
Accepted: 18 April 2023

Published online: 14 June 2023

Youngseok Kim, Andrew Eddins, Sajant Anand[3], Ken Xuan Wei[1], Ewout van den Berg[1], Sami Rosenblatt[1], Hasan Nayfeh[1], Yantao Wu[3,4], Michael Zaletel[3,5], Kristan Temme[1] & Abhinav Kandala

Quantum computing promises to offer substantial speed-ups over its classical





2008 isolated high-coherence qubits

5-qubit public quantum computer IBM (2017)

Coherence time T_2^*: $\mathbf{41.0}$, $\mathbf{43.5}$, $\mathbf{39.4}\ \mu s$

2-qubit error rate: $\mathbf{2.79\%}$, $\mathbf{2.46\%}$, $\mathbf{1.68\%}$

2023

	ibm_kyiv (127Q)				*ibm_ithaca* (65Q)			
	median	mean	min	max	median	mean	min	max
f_{01} (GHz)	4.61	4.62 ± 0.11	4.34	4.96	4.73	4.74 ± 0.10	4.54	4.93
$f_{01} - f_{12}$ (MHz)	311.07	310.77 ± 11.48	280.56	356.92	333.36	333.99 ± 5.61	316.04	355.56
T_1 (μs)	287.87	293.39 ± 84.78	85.30	567.55	183.54	180.52 ± 43.44	77.92	278.08
T_2 (μs)	127.49	156.73 ± 109.47	16.18	456.02	183.70	182.88 ± 91.05	19.27	407.15

What is next? 202? – 203?

Quantum error correction codes:

- Several **physical qubits** comprise a **logical qubit**
- The required **qubit number** is about **10000**
- The required qubit **error rate** $< \mathbf{0.001\%}$

Article

Evidence for the utility of quantum computing before fault tolerance

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Quantum computing promises to offer substantial speed-ups over its classical

The dynamics of a spin lattice with nearest neighbor interaction has been simulated

$$H=-J\sum_{\langle i,j\rangle} Z_iZ_j + h\sum_i X_i,$$

The exact simulation requires Trotter decomposition of evolution operator

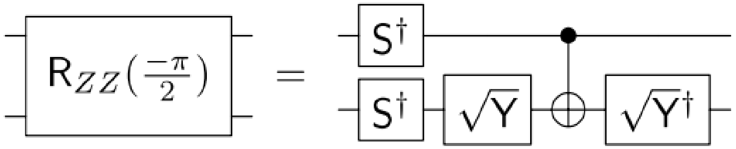
$$U(t) = \left[e^{-iH_{ZZ}\,\delta t}\, e^{-iH_X\,\delta t}\right]^N$$

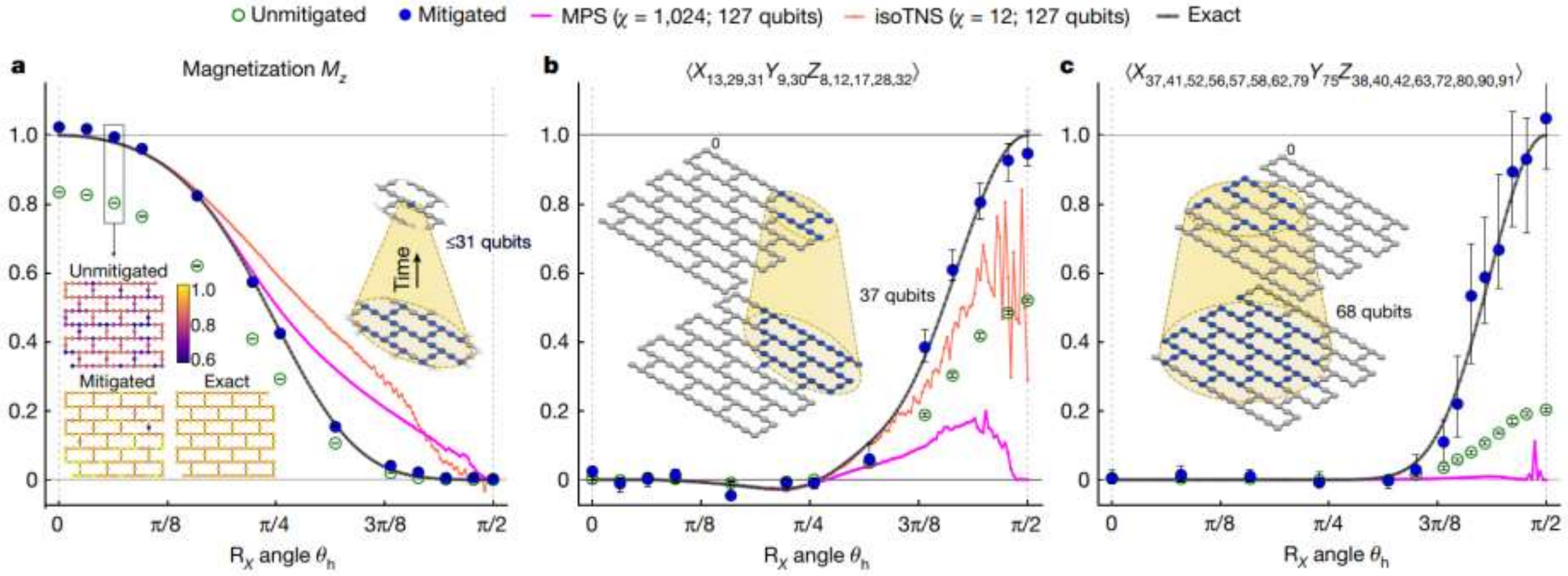
$$e^{-iH_{ZZ}\delta t} = \prod_{\langle i,j\rangle} e^{iJ\delta t Z_iZ_j} = \prod_{\langle i,j\rangle} R_{Z_iZ_j}(-2J\delta t)$$

$$e^{-iH_X\delta t} = \prod_i e^{-ih\delta t X_i} = \prod_i R_{X_i}(2h\delta t),$$

$2J\delta t \ll 1$

What has been done: $\delta t = \dfrac{\pi}{4J}$





The roadmap for practical quantum computation

Formulate a set of industry motivated computational problems:
- Continuum dynamics
- Flow dynamics

These problems are hard to solve on a classical computer !

Develop and analyse the quantum counterpart algorithm:
- Quantum scheme size (number of qubits, of quantum gates)
- Noise sensitivity (fidelity)
- Computational gain at a given noise level

Quantum hardware requirements :
- Required accuracy of quantum gates to reach quantum supremecy
- Number of qubits and scheme connectivity
- Required coherence time

Quantum Poisson Solver (QPS): direct finite difference method

$$-\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} = q(\vec{x})$$

Solution is encoded into **amplitudes** of a quantum state of a **qubit register**:

$$|f\rangle = \sum_{x,y} f(x,y)|x\rangle|y\rangle$$

$$|x_k\rangle = |b_{d-1}\, b_{d-2} \ldots b_0\rangle$$

bit string $b_{k-1} b_{k-2} \ldots b_0$ represents index k of a discrete spatial lattice

Poisson equation is solved on a **discrete spatial lattice** of $N \times N \times \cdots$ nodes.
Number of nodes quantify **complexity of the problem**

1D case

$$-\frac{f_{n+1} + f_{n-1} - 2f_n}{h^2} = q_n$$

$$x \in \{x_0, \ldots x_N\}$$

$$h = x_{n+1} - x_n$$

$$\underbrace{\begin{bmatrix} -2 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 & 0 & 0 \\ & \ldots & \ldots & & \ldots & & \ldots & \\ & \ldots & \ldots & & \ldots & & \ldots & \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 1 & -2 & 1 \end{bmatrix}}_{N = 2^d} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ . \\ . \\ f_{N-3} \\ f_{N-2} \\ f_{N-1} \end{bmatrix} = \begin{bmatrix} h^2 q_1 + f_0 \\ h^2 q_2 \\ h^2 q_3 \\ . \\ . \\ h^2 q_{N-3} \\ h^2 q_{N-2} \\ h^2 q_{N-1} + f_N \end{bmatrix}$$

The QPS reduces to the solution of a large sparse linear system

$$H\,\vec{f} = \vec{b}$$

Quantum Algorithm for Linear Systems of Equations

Aram W. Harrow, Avinatan Hassidim, and Seth Lloyd
Phys. Rev. Lett. **103**, 150502 – Published 7 October 2009

Physics See Synopsis: The quantum shortcut to a solution

$$A\,\vec{x} = \vec{b} \qquad |x\rangle = A^{-1}|b\rangle$$

The main idea of the HHL algorithm to find inverse matrix through its spectral decomposition

$$A \xrightarrow{\text{phase estimation algorithm}} A = \sum_\lambda \lambda\, |a_\lambda\rangle\langle a_\lambda| \xrightarrow{\text{approximate reciprocal function calculation}} A^{-1} = \sum_\lambda \frac{1}{\lambda}\, |a_\lambda\rangle\langle a_\lambda|$$

Quantum linear solver provides an exponential speed up for sparse linear systems $N \Rightarrow \log N$

Hidden requirements of HHL algorithm: **efficient** unitary operation generated by matrix A

$$A \Rightarrow U_A(m) = \exp(-2\pi i\, m\, A), \qquad m = 1,2,\ldots$$

What does it mean **efficient?**

$$U_A = \prod_{i=1}^{K_A} G_i \qquad K_A \sim (\log N)^p$$

1 or 2-qubit quantum gate

polylogarithmic complexity

$$\begin{bmatrix} -2 & 1 & 0 & & \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & & \\ & \vdots & & \ddots & \vdots \\ & 0 & & \cdots & 0 \end{bmatrix}$$

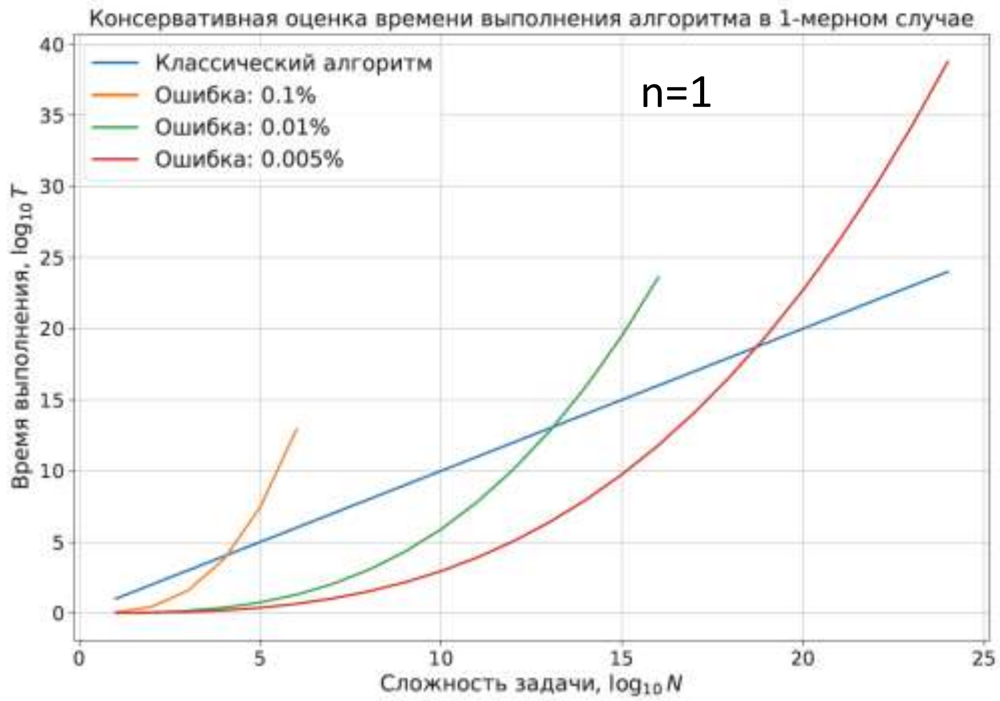
$$U_A = U_{QFT} \left(\prod_{i,j}^{d-1} e^{-i\phi_{ij}\, Z_i Z_j - i a_i\, Z_i} \right) U_{QFT}^{+}$$

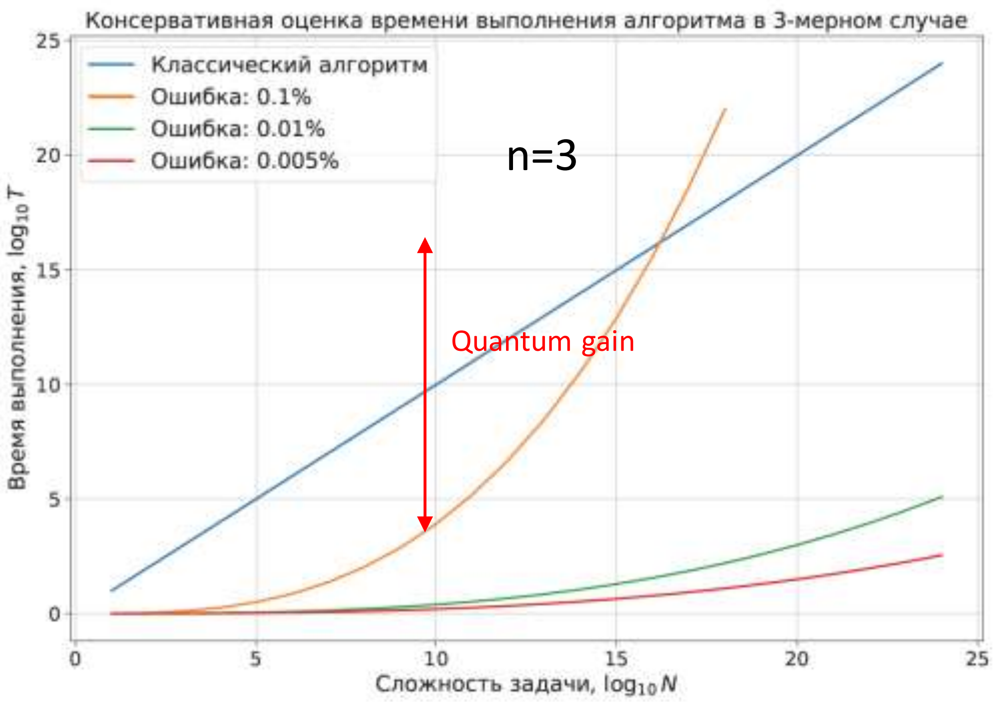
$$K_A = \frac{(\log_2 N)^2}{2} + \frac{\log_2 N}{2}$$

QPS algorithm characteristics

(for exaflop $N \sim 10^{18}$ size of computational task)

problem dimensionality	n=1	n=2	n=3
qubits number	295	355	415
number of quantum operations	775000 (2600 per qubit)	133000 (375 per qubit)	51000 (123 per qubit)
computational gain	20	10^{12}	10^{15}
required gate accuracy	99.995%	99.99%	99.98%

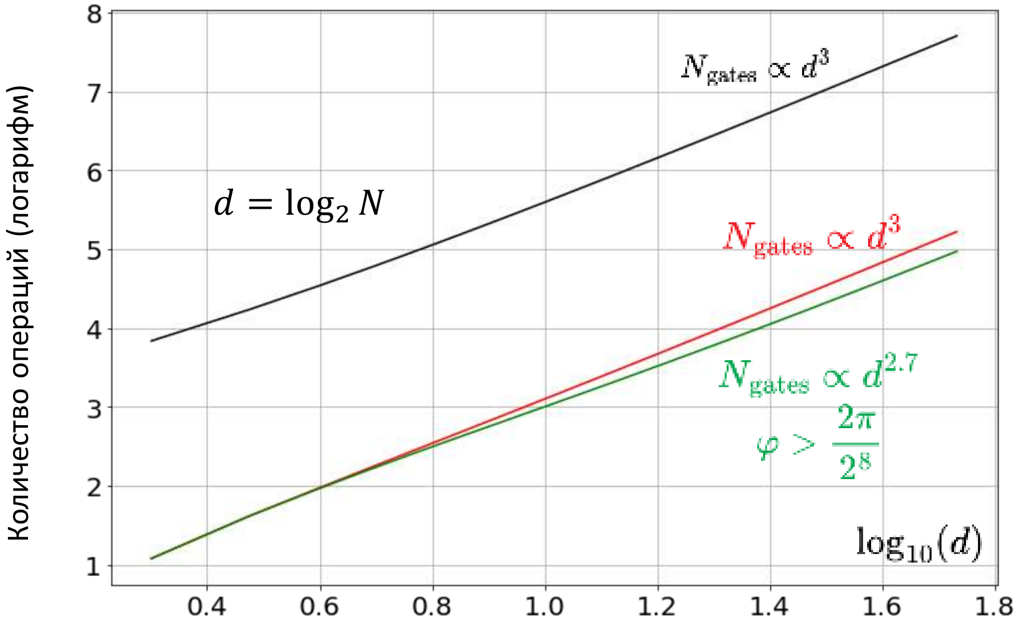




Comparison with others algorithms

Quantum fast Poisson solver:
the algorithm and complete and modular
circuit design,
Quantum Information Processing
19:170 (2020)

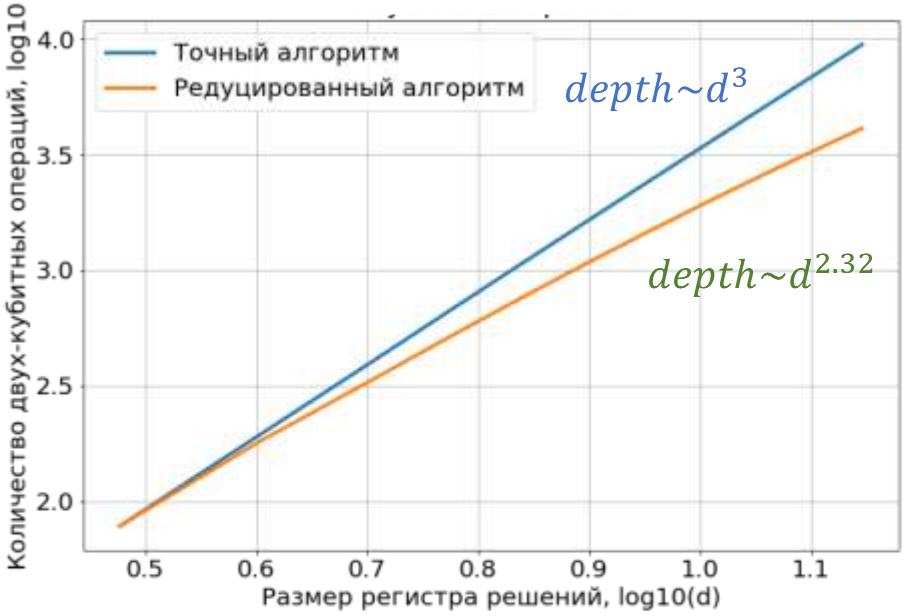
Number of gates scaling



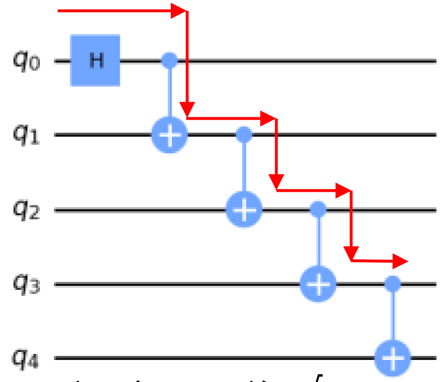
$\log_{10} \log_{10} N$

Exceeds number of atoms
in the Universe

Quantum depth of algorithm



$\log_{10} \log_{10} N$



Quantum depth: number of gates along the longest path in the circuit

depth = 5

GHZ state preparation $(|00000\rangle + |11111\rangle)/\sqrt{2}$

Main disadvantages of the direct method:

- Still requres a big qubit number $(4 + n)\log_2 N - 5$
- Number of quantum gates (quantum depth) is about 100000: **execution time exceeds** $1 - 10\ msec$
- Algorithm is **difficult to generalize** for time dynamics (for example diffusion or heat equation).
- We have **no classical information** about solution and need to repeat algorithm execution many times to infer it.

Are there any alternatives for HHL algorithm to solve linear systems?

Quantum variational solvers - QVS (main idea) $A\vec{x} = \vec{b}$

1. QVS prepares **candidates for solution** using a shallow sequence of quantum gates.
   The solution is prepared in the form of **parametrized gate sequence** acting on a reference state.

   variational ansatz

$$|x\rangle = U_K(\theta_K)U_{K-1}(\theta_{K-1}) \ldots U_j(\theta_j) \ldots U_1(\theta_1) \; |ref\ state\rangle$$

2. Measurement is performed on a candidate to evaluate its quality in terms of a **loss function**:

$$L(|x\rangle) \Rightarrow number$$

3. Optimization loop updates variational parameters to minimize the loss function with a classical optimization methods e.g. gradient descend

$$\min_{\vec{\theta}} L(|x(\vec{\theta}))$$

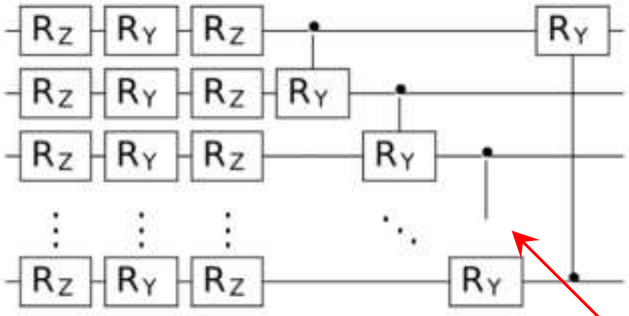
What are the possible loss functions?

$$L_R(\vec{x}) = \left\|A\,\vec{x} \; - \vec{b}\right\|^2 \qquad\qquad L_T(\vec{x}) = \left\|A\,\vec{x} \; - \vec{b}\right\|^2 + \; \|\vec{x}\|^2$$

First class of VQS: Agnostic variational ansatz (uses no information about linear system)

а)

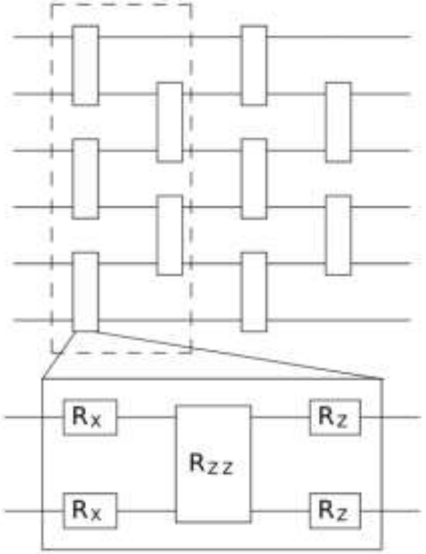
Hardware efficient



Variational parameters are encoded into single-qubit rotations at fixed entangled gates

Chess board ansatz

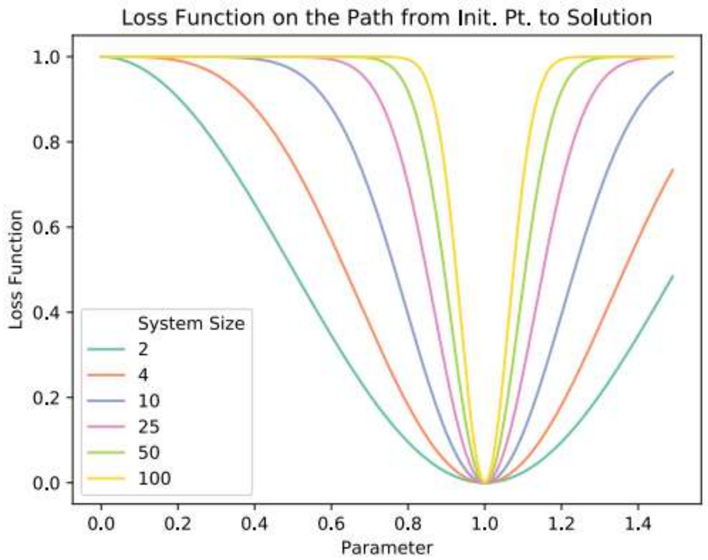
б)

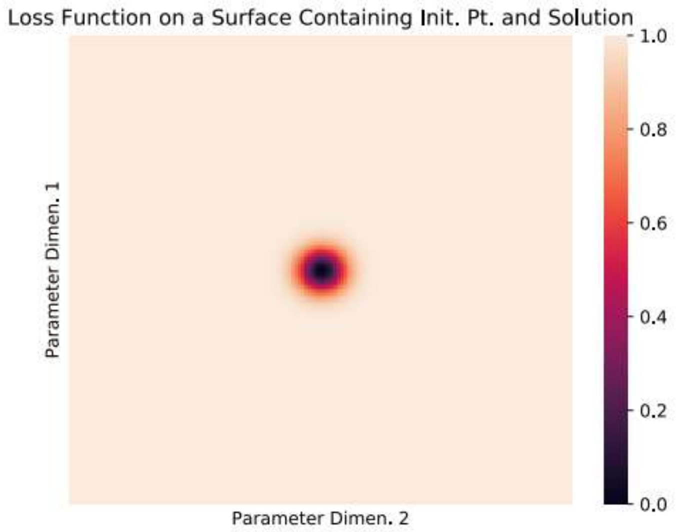


The main problem of agnostic variational solvers is an **exponentially vanishing sensitivity** to the variational parameters: “plateau effect”

$$|\partial_\theta L| \le \mathcal{O}(1/2^n)$$

The Hilbert space is huge and probe state hardly overlaps with the solution





Second class of VQS: Classical combination of variational quantum states

The main idea: different candidate states are **classically combined** to form a solution:

$$A\vec{x} = \vec{b}$$

$$|x\rangle \in Span\{\, |\psi_1\rangle, \quad \dots , \quad |\psi_K\rangle \,\}$$

candidate states

Search for the solution in the form:

$$|x\rangle = \alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle + \cdots + \alpha_K|\psi_K\rangle$$

Optimize variational parameters $\alpha_1, \alpha_2, \dots$ using classical convex optimization (polynomially hard problem)

Advantages of the method:

- The quantum state $|x\rangle$ **is never created** during algorithm execution, only simple quantum states $|\psi_k\rangle$ are generated.
- Algorithm requires the measurement of $K \times K$ simple averages:

$$L_R(|x\rangle) = \langle x|A^+A|x\rangle - 2Re\langle x|A|b\rangle + 1 \quad \Rightarrow \quad \langle\psi_i|A^+A|\psi_j\rangle \quad \langle\psi_i|A|b\rangle$$

Measured on a QC

- The loss function is calculated and optimized using classical computer

$$\sum_{i=1}^{K}\sum_{j=1}^{K}\langle\psi_i|A^+A|\psi_j\rangle$$

The main question: Where do we take good candidate states from?

Ansatz tree approach

(Huang H. Y., Bharti K., Rebentrost P. arXiv:1909.07344.)

$$A\vec{x} = \vec{b}$$

Every Hermitian matrix has a **unitary decomposition**: $A = \sum_i \beta_i \, U_i$

We make use of the form of the linear system!

Candidate states: $|\psi_{klm\ldots}\rangle = \ldots U_m \, U_l U_k |b\rangle$ are generated by unitary from decomposition of A

The variational state is built up step by step in the form of the tree:

$$|x\rangle = \alpha_0 |b\rangle + \alpha_1 \, U_1 |b\rangle + \alpha_2 \, U_2 U_1 |b\rangle \; + \cdots$$

Tree root
(**0 step**)

1 gen. child
(**1 step**)

2 gen. child
(**2 step**)

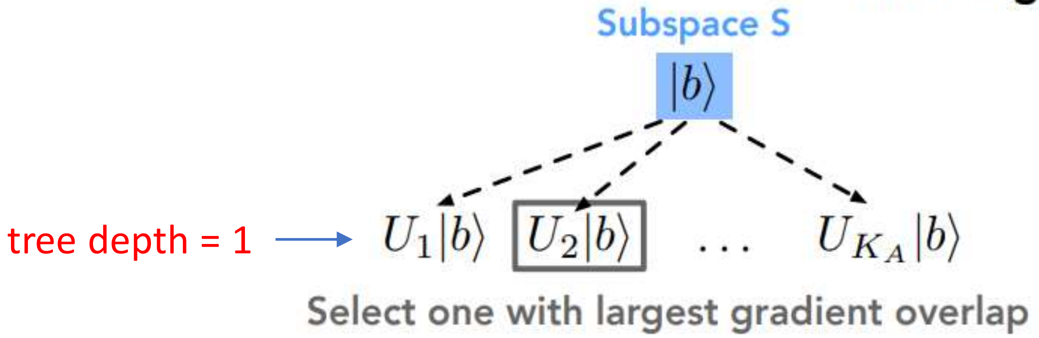
There is an efficient unitary decomposition:

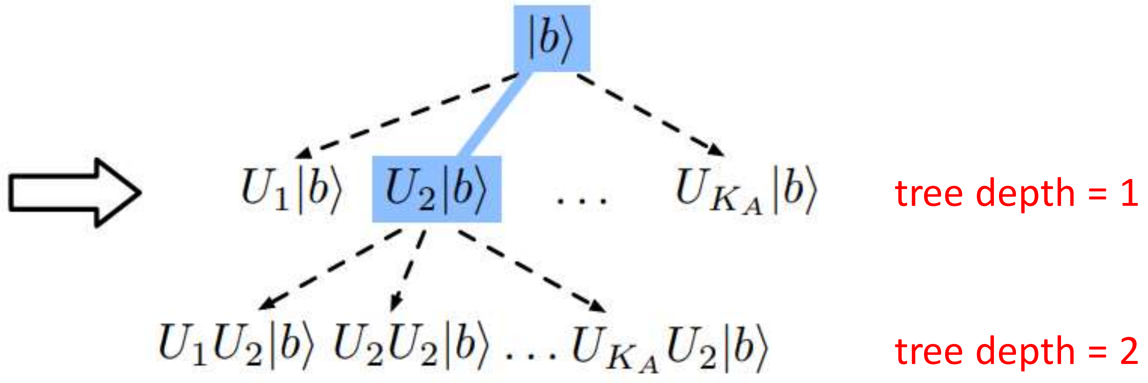
$$A = U_{QFT} \left(\sum_{k,m=0}^{d-1} d_{k,m} \, Z_k Z_m + \zeta_k Z_k \right) U_{QFT}^{+}$$

The candidate states a generated by a Pauli Z operators followed by Fourier transform

$$|\psi_k\rangle = Z_{i_1} Z_{i_2} \ldots Z_{i_k} U_{QFT} |b\rangle$$

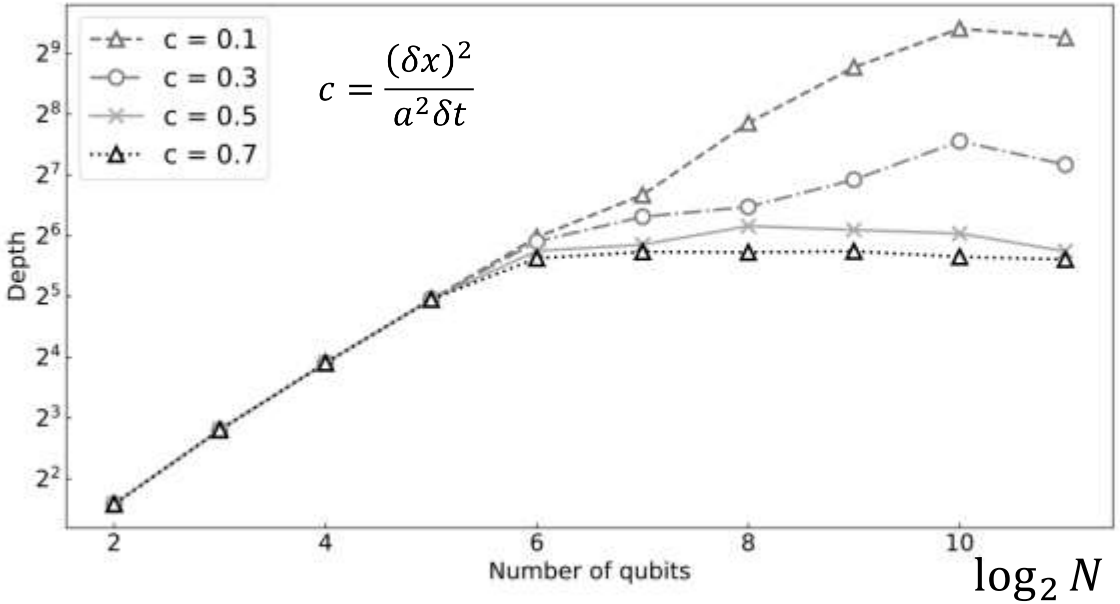
Growing the Ansatz Tree





Choose U_i which gives maximal change of the loss function among $K_A = (\log_2 N)^2/2$

The important question: How depth of the tree growths with the size of the problem?



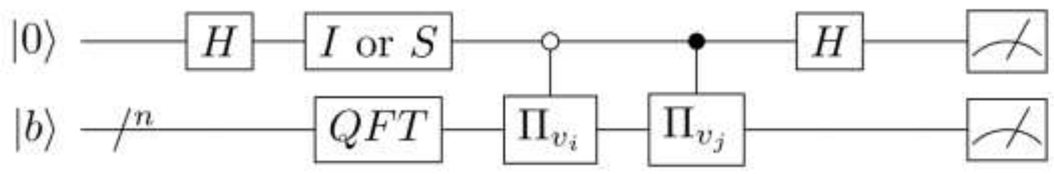
Comparison between direct and variational (Ansatz tree) methods for solving Poisson equation

1-dimensional case

	Direct solver	Ansatz tree
Qubit number	295	75
Number of gates	775000	25000 (needed to create solution)
required gate accuracy	99.995%	> 99.83%
Execution time	$(\log N)^3$	$(\log N)^{4+p}$

Main advantages: 1) during algorithm execution one uses short quantum circuits with low quantum depth

Circuit for measurement of $\langle\psi_i|A|b\rangle$



$(\log_2 N)^2/2$

$\leq 4\log_2 N$

$\Pi_{v_i} = Z_{i_1} Z_{i_2} \dots Z_{i_k}$

Quantum depth 2026 for $N = 10^{18}$

2) Solution constructed in the form of a classical combination of states and thus known classically

Ansatz tree for solution of heat equation

$$\frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial x^2} + f(x,t)$$

implicit
finite difference
scheme

$$U_{n+1}^{\tau+1} + U_{n-1}^{\tau+1} - (2+c)U_n^{\tau+1} = b_n^\tau$$

$$b_n^\tau = f_x^\tau \ \delta t + c\ U_n^\tau$$

lattice
parameter

$$c = \frac{(\delta x)^2}{a^2 \delta t}$$

On each time step algorithm solves Poisson-like equation with updated vector $\vec{b}$ $A(c)\ \vec{x} = \vec{b}$

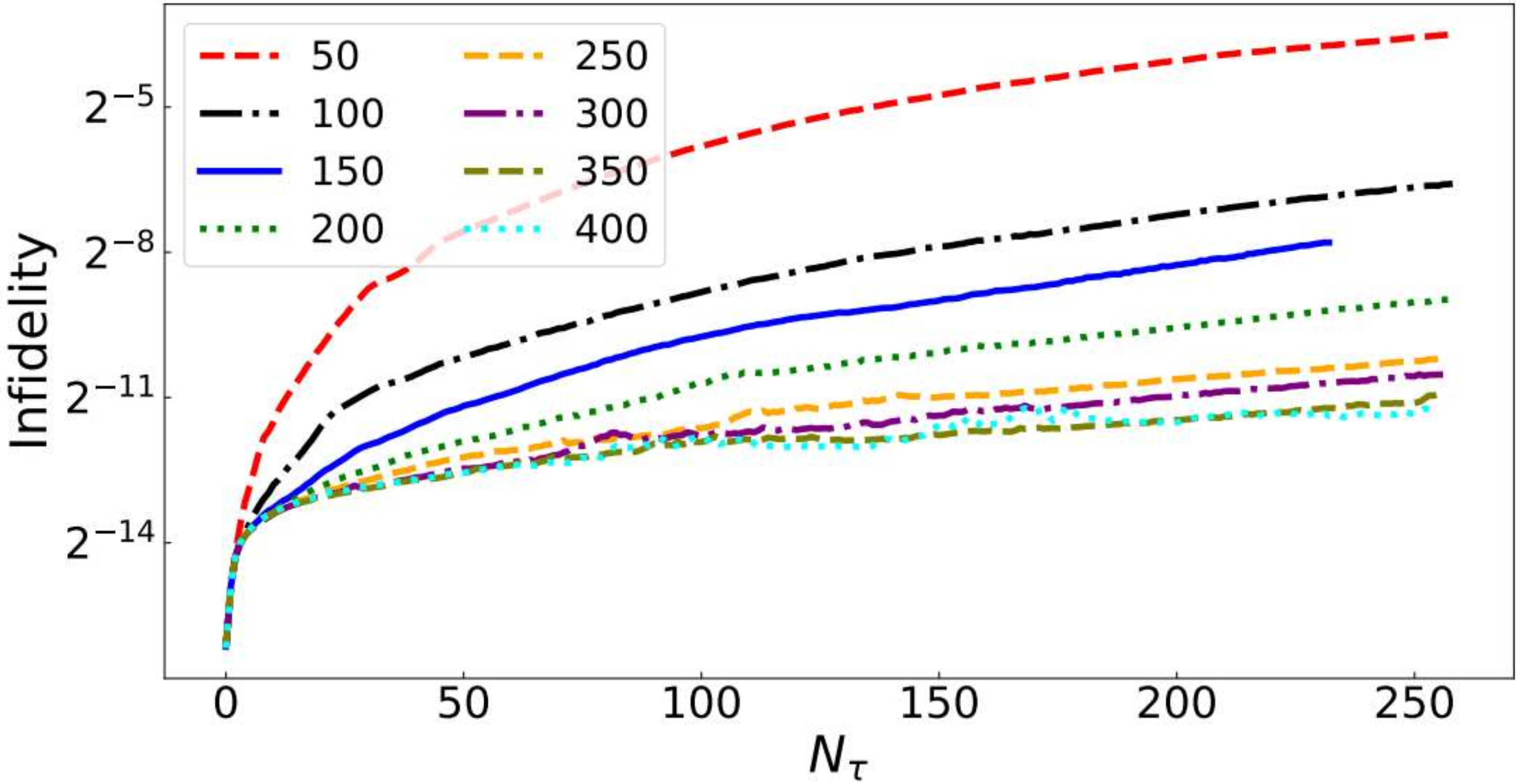
$$A(c) = \begin{pmatrix} -2-c & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2-c & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2-c & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & -2-c & 1 & 0 \\ 0 & 0 & \cdots & 1 & -2-c & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2-c \end{pmatrix}$$

Depth analysis of variational quantum algorithms for the heat equation

N. M. Guseynov, A. A. Zhukov, W. V. Pogosov, and A. V. Lebedev
Phys. Rev. A **107**, 052422 – Published 26 May 2023

Full numerical simulation of a time dependent heat equation

$N = 2^{11} = 2048$



Variational quantum Ansatz tree approach for the heat equation

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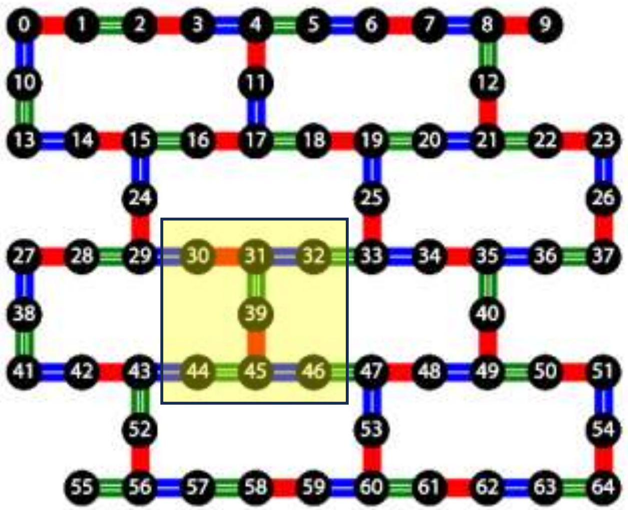
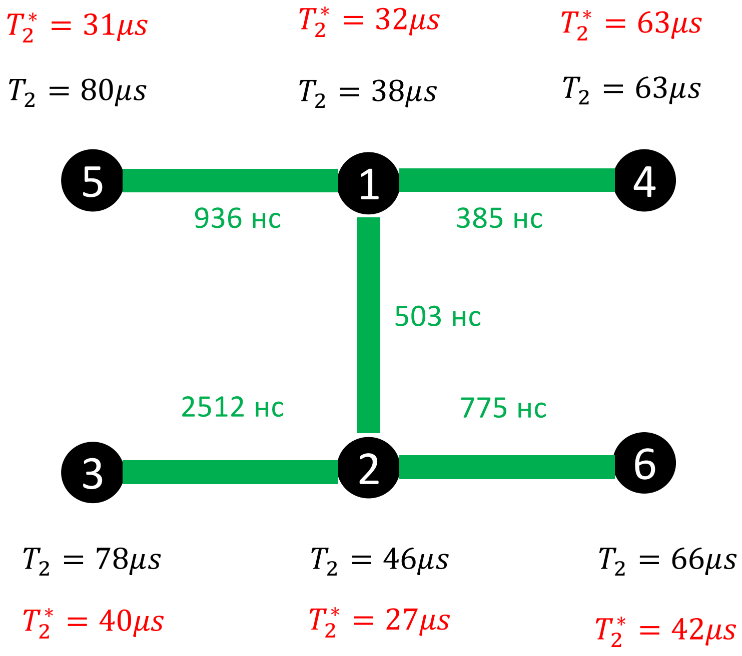
Russian-Chinese International School "Superconducting functional materials for advanced quantum technologies"

Where are we from hardware point of view





Conclusions:

- We have developed two types of algorithms for solving linear differential equations.
- For the exaflops problem size we have found the necessary requirements for the quantum processor:

	495 qubits (direct method)	75 qubit (variational solver)
error rates:	< 0.02%	< 0.1%

- We are one the route to develop a large scale quantum processor (5-year lag from leading teams)
- 6-qubit quantum processor has been deployed

Thank you for attention!